Quantum Inf Process (2016) 15:3055–3063
DOI 10.1007/s11128-016-1305-0

Mixedness and entanglement in the presence of localized closed timelike curves

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Received: 3 September 2015 / Accepted: 16 March 2016 / Published online: 30 March 2016

Abstract We examine mixedness and entanglement of the chronology-respecting (CR) system assuming that quantum mechanical closed timelike curves (CTCs) exist in nature. In order to discuss these two issues analytically, we introduce the qubit system and apply the general controlled operations between CR and CTC systems. We use the magnitude of Bloch vector as a measure of mixedness. While Deutschian-CTC (D-CTC) either preserves or decreases the magnitude, postselected-CTC (P-CTC) can increase it. Non-intuitively, even the completely mixed CR qubit can be converted into a pure state after CTC qubit travels around the P-CTC. It is also shown that while D-CTC cannot increase the entanglement of CR system, P-CTC can increase it. This means that any partially entangled state can be maximally entangled pure state if P-CTC exists. Thus, distillation of P-CTC-assisted entanglement can be easily achieved without preparing the multiple copies of the partially entangled state.

Keywords Closed timelike curves · Mixedness · Entanglement

1 Introduction

It is well known that the theory of general relativity allows the possible existence of closed timelike curves (CTCs) [1–3]. However, allowance of time travel generates logical paradoxes such as the *grandfather paradox*, in which the time traveler performs an action that causes her future self not to exist. Furthermore, the existence of CTCs cannot be compatible with a standard quantum mechanics, because quantum

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mechanics allows only unitary evolution. In order to solve these difficulties, Deutsch [4] modifies the standard quantum mechanics, which allows the non-unitary evolution in the presence of CTCs. To escape the *grandfather paradox*, in addition, he imposes the self-consistent constraint of CTC interaction (see Eq. 1). Thus, it makes it possible to explore the properties of CTCs without relying on the exotic spacetime geometries.

Then, it is natural to ask how quantum mechanics is modified if Deutsch's CTCs (D-CTCs) exist. For last few years, this question was explored in the various contexts [5–11]. Among them, most striking result is that any non-orthogonal states can be perfectly distinguished if one can access to D-CTCs [8]. This fact implies that security of usual quantum cryptography scheme such as BB84 protocol [12] is not guaranteed. Subsequently, the authors of Ref. [13] raised a question on the perfect discrimination and computational power in the presence of D-CTCs. They argued that when the input state is a labeled mixture, the assistance of CTCs in distinguishability and computational power is of no use. However, their argument was also criticized in Ref. [14]. The authors of Ref. [14] claimed by constructing the equivalent circuit that the CTCs would be a true powerful resource for quantum information processing. Another non-intuitive result arising due to existence of D-CTCs is that any arbitrary dimensional quantum states can be perfectly cloned if the dimension of the CTC system is infinite [10,11]. Thus, the well-known no-cloning theorem [15] can be broken in the presence of D-CTCs. In our opinion, however, still it seems to be open problem to determine whether or not the assistance of CTCs allows such non-intuitive results, because the debate between Ref. [8] and Ref. [13] is not concluded yet.

Mathematically, the Deutsch's self-consistency condition is expressed as

$$\rho_{\text{out}}^{(\text{CTC})} \equiv \text{tr}_{CR}\left[U\left(\rho_{\text{in}}^{(\text{CR})} \otimes \rho_{\text{in}}^{(\text{CTC})}\right)U^{\dagger}\right] = \rho_{\text{in}}^{(\text{CTC})} \tag{1}$$

where $\rho_{\text{in}}^{(\text{CR})}$ and $\rho_{\text{in}}^{(\text{CTC})}$ are input states of the chronology-respecting (CR) and chronology-violating systems, respectively. Here, $\rho^{(\text{CTC})}$ is a quantum state of system traversing the CTC and $\rho^{(\text{CR})}$ is a quantum state of system, which only interacts with $\rho^{(\text{CTC})}$, but not traversing the CTC. The operator U represents the unitary interaction between CR and CTC systems. Since the self-consistency condition imposes the equality of input and output CTC states, it naturally solves the *grandfather paradox*. Deutsch [4] showed that the fixed-point solution of Eq. (1) always exists, but it does not necessarily have to be unique. If there are many solutions, Deutsch suggested the *maximum entropy rule*. If $\rho^{(\text{CTC})}$ is fixed, the CR system is evolved as

$$\rho_{\text{in}}^{(\text{CR})} \rightarrow \rho_{\text{out}}^{(\text{CR})} \equiv \text{tr}_{CTC}\left[U\left(\rho_{\text{in}}^{(\text{CR})} \otimes \rho_{\text{in}}^{(\text{CTC})}\right)U^{\dagger}\right]. \tag{2}$$

The output state $\rho_{\text{out}}^{(\text{CR})}$ is in general non-unitary evolution of $\rho_{\text{in}}^{(\text{CR})}$, because $\rho_{\text{out}}^{(\text{CR})}$ depends on both $\rho_{\text{in}}^{(\text{CR})}$ and $\rho^{(\text{CTC})}$, and $\rho^{(\text{CTC})}$ also depends on $\rho_{\text{in}}^{(\text{CR})}$.

The postselected CTCs [16–18] (P-CTCs) are another type of quantum mechanical CTCs, which also solve the paradoxes. In P-CTC picture time travel effectively represents a quantum communication channel from the future to the past. If one chooses a quantum teleportation channel as the communication channel, P-CTCs provide a self-consistent picture of quantum mechanical time travel via postselected quantum teleportation [19]. It is based on the Horowitz–Maldacena "final state condition" [20] for black hole evaporation [21] and, unlike D-CTCs, are consistent with path-integral approaches to CTCs [22,23]. In P-CTCs formalism the state in CTC system is not explicitly specified while the output state of the CR system is given by

$$\rho_{\text{out}}^{(\text{CR})} \propto V \rho_{\text{in}}^{(\text{CR})} V^{\dagger} \tag{3}$$

where $V = \text{tr}_{CTC} U$. It turned out that though P-CTCs are less powerful resource than D-CTCs in the quantum information processing, they also have a computational and discrimination power [24].

In this Letter, we explore the following issues. By introducing simple qubit system and general controlled operations, mixedness of the CR system is examined. The mixedness is measured by a magnitude of Bloch vector. It is shown that the magnitude of Bloch vector for qubit system assisted by D-CTCs either preserves or decreases. Thus, the pure CR state can propagate to mixed state when CTC qubit travels around the D-CTC. In this sense, CTC problem resembles the information loss problem [25,26] in Hawking radiation [27]. For P-CTCs, however, the magnitude of Bloch vector can increase. In this case, a mixed state can evolve to a pure state. Even the completely mixed state can be converted into a pure state if the controlled operation is chosen appropriately. We also examine how the entanglement of the CR system is changed in the presence of CTCs. While D-CTCs always either preserve or degrade the entanglement, P-CTCs can increase it. Thus, if any partially entangled CR state is prepared, one can change it into a maximally entangled pure state if P-CTCs assist. This fact implies that distillation of entanglement [28,29] can be easily achieved without preparing the many copies of the partially entangled state if P-CTCs are appropriately exploited.

2 Mixedness in the presence of CTCs

Since both D-CTC and P-CTC allow the non-unitary evolution for CR system, it is natural to ask how the mixedness is changed in the presence of the CTCs. It is known that single-qubit state is pure or completely mixed when the magnitude of Bloch vector is one or zero. Thus, it is natural to choose the magnitude of Bloch vector as a measure of mixedness.

The most general interaction between single-qubit CR and CTC systems is unitary group $U(2)$, whose generators are Pauli matrices and identity. Treating the general U_2 interaction seems to be difficult because it has too many free parameters. Since we want to discuss the mixedness and entanglement issues on the analytic ground, we want to choose more simple interaction. Since the controlled gate is frequently used in the quantum information processing, we choose the controlled-U_2 interaction between CR and CTC systems. In this case, U_2 is represented by four real parameters as follows;

$$U_2 = e^{i\phi/2} \begin{pmatrix} \cos\theta e^{i\phi_1} & \sin\theta e^{i\phi_2} \\ -\sin\theta e^{-i\phi_2} & \cos\theta e^{-i\phi_1} \end{pmatrix}. \tag{4}$$

The initial CR state is chosen as a general form of one-qubit $\rho_{\text{in}}^{(\text{CR})} = \frac{1}{2}(I_2 + \boldsymbol{r}\cdot\boldsymbol{\sigma})$, where $|\boldsymbol{r}| = 0$ and $|\boldsymbol{r}| = 1$ correspond to the completely mixed and pure states, respectively. We assume $r_3 \neq 1$ because if $r_3 = 1$, the controlled operation cannot be turned on.

For the case of P-CTC, one can derive the output CR state by making use of Eq. (3) as $\rho_{\text{out}}^{CR} = \frac{1}{2}(I_2 + \boldsymbol{r}'\cdot\boldsymbol{\sigma})$, where

$$\begin{aligned} r_1' &= \frac{2\cos\theta\cos\phi_1}{(1+r_3) + \cos^2\theta\cos^2\phi_1(1-r_3)}\left(r_1\cos\frac{\phi}{2} - r_2\sin\frac{\phi}{2}\right) \\ r_2' &= \frac{2\cos\theta\cos\phi_1}{(1+r_3) + \cos^2\theta\cos^2\phi_1(1-r_3)}\left(r_1\sin\frac{\phi}{2} + r_2\cos\frac{\phi}{2}\right) \\ r_3' &= \frac{(1+r_3) - \cos^2\theta\cos^2\phi_1(1-r_3)}{(1+r_3) + \cos^2\theta\cos^2\phi_1(1-r_3)}. \end{aligned} \tag{5}$$

Then, one can show directly

$$|\boldsymbol{r}'|^2 - |\boldsymbol{r}|^2 = (1 - |\boldsymbol{r}|^2)\left[1 - \left(\frac{2\cos\theta\cos\phi_1}{(1+r_3) + \cos^2\theta\cos^2\phi_1(1-r_3)}\right)^2\right]. \tag{6}$$

As expected, Eq. (6) guarantees that the pure input CR state always evolves into pure. Since, however, the right-hand side of Eq. (6) can be positive or negative depending on U_2, the CR state can evolve with increasing or decreasing its mixedness. Even though $\rho_{\text{in}}^{(\text{CR})}$ is completely mixed state, $\rho_{\text{out}}^{(\text{CR})}$ becomes pure state $|0\rangle\langle 0|$ when $\theta = \pi/2$ or $\phi_1 = \pi/2$. Thus, P-CTC allows the evolution from mixed to pure state if qubit travels around the P-CTC.

However, the situation is different if the CR system is assisted by D-CTC. If the initial CTC state $\rho_{\text{in}}^{(\text{CTC})}$ is chosen as a general form $\rho_{\text{in}}^{(\text{CTC})} = \frac{1}{2}(I_2 + \boldsymbol{s}\cdot\boldsymbol{\sigma})$, one can show directly $\rho_{\text{out}}^{(\text{CTC})} \equiv \text{tr}_{CR}\left[U\rho_{\text{in}}^{(\text{CR})} \otimes \rho_{\text{in}}^{(\text{CTC})} U^\dagger\right] = \frac{1}{2}\left(I_2 + \boldsymbol{s}'\cdot\boldsymbol{\sigma}\right)$, where

$$\begin{aligned} \Delta s_1 = -(1-r_3)\Big[& s_1\left(\sin^2\phi_1 + \sin^2\theta\cos(\phi_1+\phi_2)\cos(\phi_1-\phi_2)\right) \\ & - s_2\left(\cos^2\theta\sin\phi_1\cos\phi_1 + \sin^2\theta\sin\phi_2\cos\phi_2\right) \\ & + s_3\sin\theta\cos\theta\cos(\phi_1+\phi_2)\Big] \\ \Delta s_2 = -(1-r_3)\Big[& s_1\left(\cos^2\theta\sin\phi_1\cos\phi_1 - \sin^2\theta\sin\phi_2\cos\phi_2\right) \\ & + s_2\left(\sin^2\phi_1 - \sin^2\theta\sin(\phi_1+\phi_2)\sin(\phi_1-\phi_2)\right) \end{aligned}$$

Table 1 Solutions of the self-consistency condition for various U_2

Condition	solution of self-consistency condition
$\sin\theta = 0, \sin\phi_1 = 0$	No constraint
$\sin\theta = 0, \sin\phi_1 \neq 0$	$s_1 = s_2 = 0$
$\sin\theta \neq 0, \sin\phi_1 = 0$	$s_1 = s_2 \tan\phi_2, s_3 = 0$
$\sin\theta \neq 0, \sin\phi_1 \neq 0$	$s_1 = s_3 \tan\theta \csc\phi_1 \sin\phi_2, s_2 = s_3 \tan\theta \csc\phi_1 \cos\phi_2$

$$- s_3 \sin\theta \cos\theta \sin(\phi_1 + \phi_2)\Big]$$

$$\Delta s_3 = (1 - r_3) \sin\theta \Big[s_1 \cos\theta \cos(\phi_1 - \phi_2) + s_2 \cos\theta \sin(\phi_1 - \phi_2) - s_3 \sin\theta \Big] \tag{7}$$

with $\Delta s_j = s'_j - s_j$ $(j = 1, 2, 3)$. Then, the self-consistency condition (1) simply reduces to $\Delta s_j = 0$.

The solutions of the self-consistency condition are summarized in Table 1 for various U_2. We will focus on the case of $\sin\theta \neq 0$ and $\sin\phi_1 \neq 0$, since the remaining ones can be discussed in a similar way. Since $|\boldsymbol{s}| \leq 1$, the self-consistency condition implies

$$s_3^2 \leq \frac{\sin^2\phi_1}{\sin^2\phi_1 + \tan^2\theta} \tag{8}$$

where equality holds for pure CTC state. Then, the output CR state becomes $\rho_{\text{out}}^{(\text{CR})} \equiv \text{tr}_{CTC}\left[U\rho_{\text{in}}^{(\text{CR})} \otimes \rho_{\text{in}}^{(\text{CTC})} U^\dagger\right] = \frac{1}{2}\left(I_2 + \boldsymbol{r}' \cdot \boldsymbol{\sigma}\right)$, where

$$r'_1 = Pr_1 - Qr_2 \quad r'_2 = Qr_1 + Pr_2 \quad r'_3 = r_3 \tag{9}$$

with

$$\begin{aligned} P &= \cos\frac{\phi}{2} \cos\theta \cos\phi_1 - s_3 \sin\frac{\phi}{2} \frac{\sin^2\theta + \cos^2\theta \sin^2\phi_1}{\cos\theta \sin\phi_1} \\ Q &= \sin\frac{\phi}{2} \cos\theta \cos\phi_1 + s_3 \cos\frac{\phi}{2} \frac{\sin^2\theta + \cos^2\theta \sin^2\phi_1}{\cos\theta \sin\phi_1}. \end{aligned} \tag{10}$$

Therefore, $|\boldsymbol{r}'|^2 = (P^2 + Q^2)(r_1^2 + r_2^2) + r_3^2$, where

$$P^2 + Q^2 = \cos^2\theta \cos^2\phi_1 + s_3^2 \left(\frac{\sin^2\theta + \cos^2\theta \sin^2\phi_1)}{\cos\theta \sin\phi_1}\right)^2.$$

When s_3^2 saturates the inequality (8), it is easy to show $|\boldsymbol{r}'| = |\boldsymbol{r}|$. Thus, the mixedness of the CR system is preserved when the CTC system is pure. When CTC state is mixed, $\rho_{\text{out}}^{(\text{CR})}$ is more mixed than $\rho_{\text{in}}^{(\text{CR})}$, i.e., $|\boldsymbol{r}'| < |\boldsymbol{r}|$. If the Deutsch's

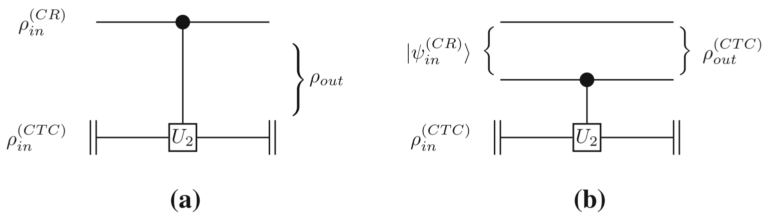

Fig. 1 **a** Circuit for examining the mixedness of CR system when CR and CTC systems interact with each other through general controlled operations. The U_2 is represented by Eq. (4). The double vertical bars on the *bottom left* and *right* indicate the past and future mouths of the wormhole for the CTC. **b** Circuit for examining the entanglement of CR state in the presence of CTC. We choose the initial CR state as a partially entangled state $|\psi_{\text{in}}^{(\text{CR})}\rangle = \alpha|00\rangle + \beta|11\rangle$ with $\alpha^2 + \beta^2 = 1$ and $|\beta| \geq |\alpha|$

maximum entropy postulate is chosen, $\rho_{\text{out}}^{(\text{CR})}$ becomes the maximal mixed state $|\boldsymbol{r}'|^2 = \cos^2\theta\cos^2\phi_1(r_1^2+r_2^2)+r_3^2$. Thus, any pure states of the form $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$ can be converted into the completely mixed state when $\cos\theta = 0$ or $\cos\phi_1 = 0$ if maximum entropy rule is chosen.

3 Entanglement in the presence of CTCs

We examine how the entanglement of CR system is changed in the presence of CTCs. To explore this issue, we introduce partially entangled two-qubit initial state $|\psi_{\text{in}}^{(\text{CR})}\rangle = \alpha|00\rangle + \beta|11\rangle$ where $\alpha^2+\beta^2 = 1$. We also choose $|\beta| \geq |\alpha|$ without loss of generality. One party of CR system interacts with CTC through the controlled-U_2 operation. The other party has no interaction with the CTC system. This situation is depicted in Fig. 1b as a quantum circuit. We will use the concurrence [30] as an entanglement measure. The concurrence of $|\psi_{\text{in}}^{(\text{CR})}\rangle$ is $2|\alpha\beta|$.

For the case of P-CTC, one can derive $\rho_{\text{out}}^{(\text{CR})}$ by making use of Eq. (3) in a form

$$\rho_{\text{out}}^{(\text{CR})} = \frac{1}{\alpha^2 + \beta^2\cos^2\theta\cos^2\phi_1}\Big[\alpha^2|00\rangle\langle 00| + \beta^2\cos^2\theta\cos^2\phi_1|11\rangle\langle 11| + \alpha\beta e^{-i\phi/2}\cos\theta\cos\phi_1|00\rangle\langle 11| + \alpha\beta e^{i\phi/2}\cos\theta\cos\phi_1|11\rangle\langle 00|\Big]. \quad (11)$$

The concurrence of $\rho_{\text{out}}^{(\text{CR})}$ is easily computed by following the procedure of Ref. [30], and final expression is

$$\mathcal{C}\left(\rho_{\text{out}}^{(\text{CR})}\right) = 2|\alpha\beta|\gamma \quad (12)$$

where the ratio γ is

$$\gamma = \frac{|\cos\theta\cos\phi_1|}{\alpha^2 + \beta^2\cos^2\theta\cos^2\phi_1}. \quad (13)$$

It is remarkable to note that the ratio γ is dependent on both U_2 and the initial CR state. Surprisingly, one can always make $\rho_{\text{out}}^{(\text{CR})}$ maximally entangled pure state $\frac{1}{\sqrt{2}}(|00\rangle \pm e^{i\phi/2}|11\rangle)$ by choosing $\cos\theta\cos\phi_1 = \pm\frac{\alpha}{\beta}$. Thus, if P-CTC exists, the distillation of entanglement of CR system can be easily achieved without preparing multiple copies of the partially entangled state. It is sufficient to prepare a single copy for complete distillation by choosing U_2 appropriately.

The situation is different for the case of D-CTC. We define the initial CTC state as a one-qubit general form $\rho_{\text{in}}^{(\text{CTC})} = \frac{1}{2}(I_2 + \boldsymbol{s}\cdot\boldsymbol{\sigma})$. Then, the output CTC state becomes $\rho_{\text{out}}^{(\text{CTC})} \equiv \text{tr}_{CR}\left[U|\psi_{\text{in}}^{(\text{CR})}\rangle\langle\psi_{\text{in}}^{(\text{CR})}|\otimes\rho_{\text{in}}^{(\text{CTC})}U^\dagger\right] = \frac{1}{2}\left(I_2 + \boldsymbol{s}'\cdot\boldsymbol{\sigma}\right)$, where Δs_j $(j = 1, 2, 3)$ is exactly the same with Eq. (7) if $1 - r_3$ is changed into $2\beta^2$. Thus, the solutions of the self-consistency condition are identical with those given in Table 1. One can also show directly that the output CR state is

$$\begin{aligned}\rho_{\text{out}}^{(\text{CR})} &\equiv \text{tr}_{CTC}\left[U|\psi_{\text{in}}^{(\text{CR})}\rangle\langle\psi_{\text{in}}^{(\text{CR})}|\otimes\rho_{\text{in}}^{(\text{CTC})}U^\dagger\right]\\ &= \alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11| + A|00\rangle\langle 11| + A^*|11\rangle\langle 00|\end{aligned} \tag{14}$$

where

$$A = e^{-i\phi/2}\alpha\beta\left[\cos\theta\cos\phi_1 - i\left(s_1\sin\theta\sin\phi_2 + s_2\sin\theta\cos\phi_2 + s_3\cos\theta\sin\phi_1\right)\right]. \tag{15}$$

It is easy to show that the concurrence of $\rho_{\text{out}}^{(\text{CR})}$ is

$$\mathcal{C}\left(\rho_{\text{out}}^{(\text{CR})}\right) = 2\min\left(|A|, |\alpha\beta|\right). \tag{16}$$

Thus, D-CTC can either preserve or decrease the entanglement of CR system.

For example, let us consider the case of $\sin\theta \neq 0$ and $\sin\phi_1 \neq 0$. Then, the variation of entanglement $\Delta\mathcal{E} \equiv \mathcal{C}\left(\rho_{\text{in}}^{(\text{CR})}\right) - \mathcal{C}\left(\rho_{\text{out}}^{(\text{CR})}\right)$ can be computed by making use of Eq. (16) and Table 1:

$$\Delta\mathcal{E} = 2|\alpha\beta|\left[1 - \sqrt{1 - \left(1 - \frac{\sin^2\phi_1 + \tan^2\theta}{\sin^2\phi_1}s_3^2\right)\left(\sin^2\theta + \cos^2\theta\sin^2\phi_1\right)}\right]. \tag{17}$$

Thus, if the inequality (8) is saturated, $\Delta\mathcal{E}$ vanishes. This means that if the CTC-state is pure, the entanglement of CR state is preserved. If we choose the maximal entropy CTC state as Deutsch suggested, the maximal degradation of entanglement $\Delta\mathcal{E} = 2|\alpha\beta|(1 - |\cos\theta\cos\phi_1|)$ occurs.

4 Conclusions

Although the theory of general relativity does allow CTC as a solution of Einstein field equations, still there are a lot of controversial for existence of CTCs. In this Letter, we have addressed two issues, mixedness and entanglement for CR system assuming that D-CTC and/or P-CTC exist(s) in nature. It was shown that while D-CTC-assisted qubit cannot increase the magnitude of its Bloch vector, P-CTC-assisted qubit can. As a result, the mixed CR state can evolve to pure CR state if P-CTC exists. Even the completely mixed state can evolve to pure state if we choose the phase angles of U_2 appropriately.

Although the CTC state is not specified explicitly for the case of P-CTC, one can get some information of P-CTC state if exists any. Let us imagine a closed systems composed by CR and P-CTC subsystems. Let us assume that they interact with each other through some unitary operation. If one uses the subadditivity of the von Neumann entropy, one can show $\Delta S^{(CTC)} \geq -\Delta S^{(CR)}$, where S is a von Neumann entropy and $\Delta S^{(\cdot)} \equiv S(\rho_{out}^{(\cdot)}) - S(\rho_{in}^{(\cdot)})$. Thus, computing the entropy difference in CR subsystem one can compute the lower bound of $\Delta S^{(CTC)}$ although we do not know the P-CTC state explicitly.

We also have studied the case where the CR system consists of bipartite partially entangled particles and one of them interacts with CTC system through controlled-U_2 operation. For the case of P-CTC surprisingly the partially entangled state can always be converted into the maximally entangled pure state if the phase angles of U_2 are chosen appropriately. If, therefore, P-CTCs exist, the distillation protocol of entanglement is easily achieved without preparing the multiple copies of the partially entangled state. For the case of D-CTC, such a non-intuitive effect disappears because D-CTC either preserves or decreases the entanglement of CR system.

There are a lot of questions in the context of CTCs. How to incorporate the general relativistic CTCs into the quantum mechanical CTCs or vice versa? What happens to the uncertainty relations if CTCs exist [11,31,32]? While P-CTC allows only the evolution of pure state to pure state, D-CTC allows the evolution of pure state to mixed state. Thus, existence of D-CTC may provide clue for the information loss problem of black hole. In this context, the information loss problem was discussed in Ref. [27] by introducing a simple toy model. Of course, rigorous and explicit analysis should be addressed to clarify a connection between existence of CTCs and information loss problem. Probably, the theory of quantum gravity may give some answers in the future.

Another interesting issue is to check whether our results of mixedness and entanglement in the presence of CTCs are valid for general interaction $U(2)$ or not. We hope to study this issue in the future.

Acknowledgments This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0011971).

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