Concurrence-based Entanglement Measure For True 4-way Entanglement

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Abstract

An entanglement monotone, which is invariant under the determinant 1 SLOCC operations and measures the true quadripartite entanglement, is explicitly constructed.

Recently, much attention is being paid to quantum technology[1]. Most important notion in quantum technology is a quantum correlation, which is usually represented by entanglement[2] of given quantum states. As shown for last two decades it plays a central role in quantum teleportation[3], superdense coding[4], quantum cloning[5], and quantum cryptography[6]. It is also quantum entanglement, which makes the quantum computer outperform the classical one[7]. Thus, it is very important to understand how to quantify and how to characterize the entanglement.

For bipartite quantum system many entanglement measures were constructed before such as distillable entanglement[8], entanglement of formation (EoF)[8], and relative entropy of entanglement (REE)[9, 10]. Especially, for two-qubit system, EoF is expressed as[11]

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right) \tag{1}$$

where $h(x)$ is a binary entropy function $h(x) = -x\ln x - (1-x)\ln(1-x)$ and C is called the concurrence. For two-qubit pure state $|\psi\rangle = \psi_{ij}|ij\rangle$ with $(i,j=0,1)$, C is given by

$$C = |\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\psi_{i_1 j_1}\psi_{i_2 j_2}| = 2|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}| \tag{2}$$

where the Einstein convention is understood and $\epsilon_{\mu\nu}$ is an antisymmetric tensor.

Although quantification of the entanglement is important, it is equally important to classify the entanglement, i.e., to classify the quantum states into the same type of entanglement. The most popular classification scheme is a classification through a stochastic local operation and classical communication (SLOCC)[12]. If $|\psi\rangle$ and $|\phi\rangle$ are in same SLOCC class, this means that $|\psi\rangle$ and $|\phi\rangle$ can be used to implement same task of quantum information theory although the probability of success for this task is different. Mathematically, if two n-party states $|\psi\rangle$ and $|\phi\rangle$ are in the same SLOCC class, they are related to each other by $|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n|\phi\rangle$ with $\{A_j\}$ being arbitrary invertible local operators[1]. However, it is more useful to restrict ourselves to SLOCC transformation where all $\{A_j\}$ belong to SL(2, C), the group of 2×2 complex matrices having determinant equal to 1. In the three-qubit pure-state system it was shown[13] that there are six different SLOCC classes, fully-separable, three bi-separable, W, and Greenberger-Horne-Zeilinger (GHZ) classes.

Classification through the SLOCC transformation enables us to construct the entanglement measures. As Ref.[14] showed, any linearly homogeneous positive function of a

[1] For complete proof on the connection between SLOCC and local operations see Appendix A of Ref.[13].

pure state that is invariant under determinant 1 SLOCC operations is an entanglement monotone. One can show that C in Eq. (2) is such an entanglement monotone as follows. Let $|\psi\rangle = \psi_{ij}|ij\rangle$ with $i, j = 0, 1$. Then, $|\tilde{\psi}\rangle \equiv (A \otimes B)|\psi\rangle = \tilde{\psi}_{ij}|ij\rangle$, where $\tilde{\psi}_{ij} = \psi_{\alpha\beta} A_{i\alpha} B_{j\beta}$. Using $\epsilon_{ij} M_{i\alpha} M_{j\beta} = (\det M)\epsilon_{\alpha\beta}$ for arbitrary matrix M, it is easy to show $\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\tilde{\psi}_{i_1 j_1}\tilde{\psi}_{i_2 j_2} = (\det A)(\det B)\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\psi_{i_1 j_1}\psi_{i_2 j_2}$, which implies that C is invariant under determinant 1 SLOCC operations.

This theorem in Ref.[14] can be applied to the three-qubit system. If $|\psi\rangle = \psi_{ijk}|ijk\rangle$, the invariant monotone is

$$\tau_3 = \left| 2\epsilon_{i_1 i_2}\epsilon_{i_3 i_4}\epsilon_{j_1 j_2}\epsilon_{j_3 j_4}\epsilon_{k_1 k_3}\epsilon_{k_2 k_4}\psi_{i_1 j_1 k_1}\psi_{i_2 j_2 k_2}\psi_{i_3 j_3 k_3}\psi_{i_4 j_4 k_4} \right|^{1/2}. \tag{3}$$

This is exactly identical with a square root of the residual entanglement[2] introduced in Ref.[15]. The three-tangle (3) has following properties. If $|\psi\rangle$ is a fully-separable or partially-separable state, its three-tangle completely vanishes. Thus, τ_3 measures the genuine 3-way entanglement. For 3-way entanglement it gives $\tau_3(\mathrm{GHZ}_3) = 1$ and $\tau_3(\mathrm{W}_3) = 0$, where

$$|\mathrm{GHZ}_3\rangle \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \qquad\qquad |\mathrm{W}_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{4}$$

For mixed state quantification of the entanglement is usually defined via a convex-roof method[8, 16]. Although the concurrence for an arbitrary two-qubit mixed state can be, in principle, computed following the procedure introduced in Ref.[11], still we do not know how to compute the three-tangle (or residual entanglement) for an arbitrary three-qubit mixed state[3]. However, the residual entanglement for several special mixtures were computed in Ref.[17]. More recently, the three-tangle for all GHZ-symmetric states[18] was computed analytically[19].

It is also possible to construct the concurrence-based monotones in the higher-qubit systems. In the higher-qubit systems, however, there are many independent monotones because the number of independent SLOCC-invariant monotones is equal to the degrees of freedom of pure quantum state minus the degrees of freedom induced by the determinant 1 SLOCC operations. For example, there are $2(2^n - 1) - 6n$ independent monotones in n-qubit system. Thus, there are 6 independent concurrence-based monotones in four-qubit system.

[2] In this paper we will call τ_3 as a three-tangle and τ_3^2 as a residual entanglement.
[3] Even in two-qubit system still we do not know how to compute the REE for arbitrary mixed states.

If $|\psi\rangle = \psi_{ijk\ell}|ijk\ell\rangle$ with $i, j, k, \ell = 0, 1$, following two concurrence-based monotones were presented in Ref.[14];

$$\tau_{4,1} = \left| \epsilon_{i_1 i_2} \epsilon_{j_1 j_2} \epsilon_{k_1 k_2} \epsilon_{\ell_1 \ell_2} \psi_{i_1 j_1 k_1 \ell_1} \psi_{i_2 j_2 k_2 \ell_2} \right| \tag{5}$$

$$\tau_{4,2} = \left| 2 \epsilon_{i_1 i_2} \epsilon_{i_3 i_4} \epsilon_{j_1 j_3} \epsilon_{j_2 j_4} \epsilon_{k_1 k_3} \epsilon_{k_2 k_4} \epsilon_{\ell_1 \ell_2} \epsilon_{\ell_3 \ell_4} \psi_{i_1 j_1 k_1 \ell_1} \psi_{i_2 j_2 k_2 \ell_2} \psi_{i_3 j_3 k_3 \ell_3} \psi_{i_4 j_4 k_4 \ell_4} \right|^{1/2}.$$

Other four more independent entanglement monotones can be obtained by including more factors of $\psi_{ijk\ell}$. As expected $\tau_{4,1}(\mathrm{GHZ}_4) = \tau_{4,2}(\mathrm{GHZ}_4) = 1$ and $\tau_{4,1}(\mathrm{W}_4) = \tau_{4,2}(\mathrm{W}_4) = 0$, where

$$|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \qquad |\mathrm{W}_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle). \tag{6}$$

However, there is a striking difference between $\tau_{4,j}$ $(j = 1, 2)$ and three-tangle. While τ_3 vanishes for partially entangled state, $\tau_{4,1}$ and $\tau_{4,2}$ do not completely vanish for some cases. For example, for $|\mathrm{BB}\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle) \otimes (1/\sqrt{2})(|00\rangle + |11\rangle)$ $\tau_{4,1}$ and $\tau_{4,2}$ become

$$\tau_{4,1}(BB) = 1 \qquad \tau_{4,2}(BB) = \frac{1}{\sqrt{2}}. \tag{7}$$

This is mainly due to the fact that $|\mathrm{BB}\rangle$ is a normal form[14, 20] in four-qubit system. In this sense, $\tau_{4,1}$ and $\tau_{4.2}$ cannot measure the genuine 4-way entanglement.

Is there a concurrence-based entanglement monotone, which vanishes for all partially entangled four qubit states and gives maximal value for the maximal entangled state $|\mathrm{GHZ}_4\rangle$? Such entanglement monotones exist and the simplest one is

$$\tau_{4,3} = \left| \epsilon_{i_1 i_2} \epsilon_{i_3 i_4} (\epsilon_{j_1 j_3} \epsilon_{j_2 j_4} + \epsilon_{j_1 j_4} \epsilon_{j_2 j_3})(\epsilon_{k_1 k_3} \epsilon_{k_2 k_4} + \epsilon_{k_1 k_4} \epsilon_{k_2 k_3}) \epsilon_{\ell_1 \ell_2} \epsilon_{\ell_3 \ell_4} \psi_{i_1 j_1 k_1 \ell_1} \cdots \psi_{i_4 j_4 k_4 \ell_4} \right|^{1/2}. \tag{8}$$

Using a formula $\epsilon_{i_1 i_2 \cdots i_N} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_N j_N} = (\det M) \epsilon_{j_1 j_2 \cdots j_N}$ where $\epsilon_{i_1 i_2 \cdots i_N}$ is a completely antisymmetric tensor, it is easy to show that $\tau_{4,3}$ is invariant under the determinant 1 SLOCC operations. Furthermore, it is straightforward to show

$$\tau_{4,3}(\mathrm{GHZ}_4) = 1 \qquad \tau_{4,3}(\mathrm{W}_4) = 0 \qquad \tau_{4,3}(\mathrm{BB}) = 0. \tag{9}$$

In order to confirm that $\tau_{4,3}$ vanishes for all partially entangled states, let us consider the following general partially entangled states

$$|\varphi_{2\otimes 2}\rangle_{ABCD} = (a_{ij}|ij\rangle)_{\Gamma_1 \Gamma_2} \otimes (b_{k\ell}|k\ell\rangle)_{\Gamma_3 \Gamma_4} \tag{10}$$

$$|\varphi_{3\otimes 1}\rangle_{ABCD} = (a_i|i\rangle)_{\Gamma_1} (b_{jk\ell}|jk\ell\rangle)_{\Gamma_2 \Gamma_3 \Gamma_4}$$

where Γ_i denotes any party in $\{A, B, C, D\}$. It is possible to show $\tau_{4,1}(\varphi_{2\otimes2}) = \sqrt{2}\tau_{4,2}(\varphi_{2\otimes2}) = 4|(a_{00}a_{11} - a_{01}a_{10})(b_{00}b_{11} - b_{01}b_{10})|$ and $\tau_{4,1}(\varphi_{3\otimes1}) = \tau_{4,2}(\varphi_{3\otimes1}) = 0$. Thus, $\tau_{4,1}$ and $\tau_{4,2}$ can be nonzero for partially entangled $2 \otimes 2$ states. However, one can show $\tau_{4,3}(\varphi_{2\otimes2}) = \tau_{4,3}(\varphi_{3\otimes1}) = 0$. Therefore, this fact with Eq. (9) guarantees that $\tau_{4,3}$ measures the genuine quadripartite entanglement.

SLOCC	$\tau_{4,1}$	$\tau_{4,2}$	$\tau_{4,3}$
L_{abc_2}	$\frac{\lvert a^2+b^2+2c^2\rvert}{1+\lvert a\rvert^2+\lvert b\rvert^2+2\lvert c\rvert^2}$	$\frac{\lvert a^4+6a^2b^2+b^4+4c^2\{2c^2+3(a-b)^2\}\rvert^{1/2}}{\sqrt{2}(1+\lvert a\rvert^2+\lvert b\rvert^2+2\lvert c\rvert^2)}$	$\frac{2\lvert (c^2-ab)^2+2c^2(a-b)^2\rvert^{1/2}}{1+\lvert a\rvert^2+\lvert b\rvert^2+2\lvert c\rvert^2}$
$L_{a_2b_2}$	$\frac{\lvert a^2+b^2\rvert}{1+\lvert a\rvert^2+\lvert b\rvert^2}$	$\frac{\lvert a^4+b^4\rvert^{1/2}}{1+\lvert a\rvert^2+\lvert b\rvert^2}$	$\frac{\lvert a^2-b^2\rvert}{1+\lvert a\rvert^2+\lvert b\rvert^2}$
L_{ab_3}	$\frac{\lvert 3a^2+b^2\rvert}{2+3\lvert a\rvert^2+\lvert b\rvert^2}$	$\frac{\lvert 12a^2(a-b)^2+(3a^2+b^2)^2\rvert^{1/2}}{\sqrt{2}(2+3\lvert a\rvert^2+\lvert b\rvert^2)}$	$\frac{2\sqrt{3}\lvert a\rvert\lvert a-b\rvert}{2+3\lvert a\rvert^2+\lvert b\rvert^2}$
L_{a_4}	$\frac{4\lvert a\rvert^2}{3+4\lvert a\rvert^2}$	$\frac{2\sqrt{2}\lvert a\rvert^2}{3+4\lvert a\rvert^2}$	0
$L_{a_2 0_{3\oplus\bar{1}}}$	$\frac{2\lvert a\rvert^2}{3+2\lvert a\rvert^2}$	$\frac{2\lvert a\rvert^2}{3+2\lvert a\rvert^2}$	$\frac{2\lvert a\rvert^2}{3+2\lvert a\rvert^2}$
$L_{0_{5\oplus\bar{3}}}$	0	0	0
$L_{0_{7\oplus\bar{1}}}$	0	0	0
$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$	0	0	0

Table I: Four-tangles $\tau_{4,1}$, $\tau_{4,2}$, and $\tau_{4,3}$ for various SLOCC equivalent classes

For completeness let us consider the G_{abcd} class in the SLOCC classification of four-qubit pure-state system introduced in Ref.[21][4] ;

$$G_{abcd} = \frac{1}{\sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}} \left[\frac{a+d}{2} (|0000\rangle + |1111\rangle) + \frac{a-d}{2} (|0011\rangle + |1100\rangle) + \frac{b+c}{2} (|0101\rangle + |1010\rangle) + \frac{b-c}{2} (|0110\rangle + |1001\rangle) \right] \tag{11}$$

where the parameters a, b, c, and d are complex numbers with nonnegative real part. Among nine SLOCC classes G_{abcd} is special in the sense that it is set of normal states[14, 20], i.e., all local states are completely mixed. Moreover, it involves the maximally entangled state $|\text{GHZ}_4\rangle$ when $(a = d = 1, b = c = 0)$ and two EPR pairs when $(a = 1, b = c = d = 0)$ or

[4] The SLOCC classification in four-qubit pure-state system was discussed in several more papers[22]. Unlike, however, two- and three-qubit cases the results of Ref.[21, 22] seem to be contradictory with each other. Although some people asserts that this contradiction is mainly due to the different approach, we think still our understanding on the four-qubit entanglement is incomplete.

$a = b = c = d = 1$. The four-tangles $\tau_{4,1}$ and $\tau_{4,2}$ for G_{abcd} are

$$\tau_{4,1} = \frac{|a^2 + b^2 + c^2 + d^2|}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \tag{12}$$
$$\tau_{4,2} = \frac{|(a^2 + b^2 + c^2 + d^2)^2 + 4\{(ab - cd)^2 + (ac - bd)^2 + (ad - bc)^2\}|^{1/2}}{\sqrt{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2)}.$$

Using Eq. (12) it is easy to reproduce Eq. (7). Especially, from the aspect of $\tau_{4,1}$ all states in G_{abcd} class are maximally entangled provided that a, b, c, and d are real. The four-tangle $\tau_{4,3}$ for G_{abcd} is

$$\tau_{4,3} = \frac{2|(ab - cd)^2 + (ac - bd)^2 + (ad - bc)^2|^{1/2}}{|a|^2 + |b|^2 + |c|^2 + |d|^2}. \tag{13}$$

Using Eq. (13) it is easy to show that $\tau_{4,3}$ for all two EPR pairs vanishes as expected. The four-tangles $\tau_{4,1}$, $\tau_{4,2}$, and $\tau_{4,3}$ for other SLOCC classes are summarized in Table I.

In this short note we construct an concurrence-based monotone, which measures the true 4-way entanglement in the qubit system. This measure can be used to quantify the quadripartite entanglement for various mixed states such as $\rho = p|\mathrm{GHZ}_4\rangle\langle\mathrm{GHZ}_4| + (1 - p)|\mathrm{BB}\rangle\langle\mathrm{BB}|$. This will be explored elsewhere.

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