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Selections of set-valued maps satisfying a linear inclusion in a single variable

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ABSTRACT

We prove that a set-valued map satisfying a linear functional inclusion in a single variable admits (in appropriate conditions) a unique selection satisfying a linear functional equation in a single variable. As a consequence there follow results on the Hyers–Ulam stability of the linear functional equation in a single variable and the equation of homomorphism (of square symmetric groupoids).

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1. Introduction

Functional inclusion is a tool for defining many notions of set-valued analysis, e.g., linear, affine, convex, concave, subadditive, superadditive set-valued maps. Finding a selection of such set-valued maps, with some special properties, is one of the main problems of set-valued analysis (see [1]). The stability theory of functional equations leads in some cases to such problems and solving them provides Hyers–Ulam stability results.

It seems to be a common conviction that the investigation of the stability of functional equations started with a result of Hyers [2], published in 1941, as a solution to a question posed by Ulam in 1940 (see [3]); let us recall the result of Hyers.

Let X be a linear normed space, Y a Banach space and $\varepsilon > 0$. Then for every function $g : X \to Y$ satisfying the inequality

$$\|g(x+y) - g(x) - g(y)\| \le \varepsilon, \quad x, y \in X, \tag{1}$$

there exists a unique additive function $f : X \to Y$ such that

$$\|g(x) - f(x)\| \le \varepsilon, \quad x \in X. \tag{2}$$

However, we are already aware of an earlier result of this kind, due to Pólya and Szegö [4, Teil I, Aufgabe 99] (cf., e.g., [5, p. 125]). For some recent examples of discussions, extensions, generalizations and critiques of the Hyers–Ulam stability we refer the reader to, e.g., [6–10].

Smajdor [11,12] and Gajda and Ger [13] observed that if g is a solution of (1), then the set-valued map $F : X \to 2^Y$ (2^Y denotes the set of all nonempty subsets of Y), given by

$$F(x) = g(x) + B(0, \varepsilon), \quad x \in X, \tag{3}$$

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where $B(0, \varepsilon)$ is the closed ball in Y centered at 0 and with the radius ε, satisfies the inclusion $F(x + y) \subseteq F(x) + F(y)$ for $x, y \in X$ (i.e., it is subadditive), and the function f, which satisfies (2), is an *additive selection* of F (i.e., $f(x+y) = f(x) + f(y)$ and $f(x) \in F(x)$ for $x, y \in X$).

Now one may ask under what conditions a subadditive set-valued map admits an additive selection. Let us recall a result of Gajda and Ger [13] (by $\delta(F(x))$ we denote the diameter of the set $F(x)$, i.e. $\delta(F(x)) := \sup_{z,w\in F(x)} \|z - w\|$).

Theorem 1.1. *Let $(S, +)$ be a commutative semigroup with zero element, X a Banach space over $\mathbb{R}$ and $F : S \to 2^X$ a set-valued map with convex and closed values such that $F(x + y) \subseteq F(x) + F(y)$ for $x, y \in S$ and $\sup_{x\in S} \delta(F(x)) < \infty$. Then F admits a unique additive selection.*

Furthermore, the previous result was extended by Nikodem and Popa to set-valued maps satisfying general linear inclusions of the form

$$F(\alpha x + \beta y + c) \subseteq \gamma F(x) + \delta F(y) + C$$
$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y + c) + C,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers, X is a real vector space, Y is a real Banach space, $F : X \to 2^Y$, $c \in X$, $C \in 2^Y$ (see [14,15]), and to set-valued maps satisfying a functional inclusion on square symmetric groupoids (see [16,17]).

It is interesting that, once we have obtained a result of Gajda–Ger type (or the extensions mentioned above), we can prove the stability of the functional equations corresponding to the functional inclusions considered (for more details see [14,16,15, 17]). Therefore the study of such problems seems to be motivated because we thus get instruments for proving Hyers–Ulam stability of functional equations. For instance we have the following simple observation.

Theorem 1.2. *Let $(S, *)$ be a groupoid, $(Z, +)$ be a commutative group, $\mathcal{F} \subseteq 2^Z$, $B \in \mathcal{F}$, and $u + B \in \mathcal{F}$ for all $u \in Z$. Suppose that every set-valued map $F : S \to \mathcal{F}$ satisfying*

$$F(x * y) \subseteq F(x) + F(y), \quad \forall x, y \in S,$$

admits a selection $f : S \to Z$ with $f(x * y) = f(x) + f(y)$ for $x, y \in S$. Then, for every function $g : S \to Z$ such that

$$g(x * y) - g(x) - g(y) \in B, \quad \forall x, y \in S, \tag{4}$$

there is a function $f : S \to Z$ with $f(x * y) = f(x) + f(y)$ for $x, y \in S$ and

$$f(x) - g(x) \in B, \quad \forall x \in S. \tag{5}$$

Proof. Define $F : S \to \mathcal{F}$ by: $F(x) = g(x) + B$ for $x \in S$. Then

$$F(x * y) = g(x * y) + B \subseteq g(x) + g(y) + B + B = F(x) + F(y)$$

for $x, y \in S$. Hence, by the hypothesis (cf. [13]), there is $f : S \to Z$ with $f(x * y) = f(x) + f(y)$ for $x, y \in S$ and $f(x) \in F(x) = g(x) + B$ for $x \in S$. □

If Z is a normed space and $B := \{x \in Z : \|x\| \le \epsilon\}$ (with a real $\epsilon \ge 0$), then (4), (5) take the forms $\sup_{x,y\in S} \|g(x * y) - g(x) - g(y)\| \le \epsilon$ and $\sup_{x\in S} \|f(x) - g(x)\| \le \epsilon$, and from Theorems 1.1 and 1.2 we can derive a generalization of the classical Hyers stability result in [2].

The goal of this paper is to obtain a result analogous to Theorem 1.1 for set-valued maps F satisfying a linear inclusion in a single variable of the form

$$a(x)F(\varphi(x)) \subseteq F(x) + \psi(x) + b(x)B,$$

where a, b, φ and ψ are given functions. This problem is obviously in connection with the Hyers–Ulam stability of the functional equation

$$a(x)f(\varphi(x)) = f(x) + \psi(x). \tag{6}$$

For more information on this functional equation and its stability see e.g. [18–26].

2. Main result

In what follows S is a nonempty set, X is a Banach space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a : S \to \mathbb{K}$, $b : S \to [0, \infty)$, $\varphi : S \to S$, $\psi : S \to X$ are given functions, and $B \in 2^X$ is balanced, convex and with $\delta(B) < \infty$. By φ^j we denote the j-th iterate of φ for $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (where $\varphi^0(x) = x$ for $x \in S$). For every $\lambda \in \mathbb{K}$ and $A, D \in 2^X$ we write $A + D := \{a + b \mid a \in A,\ b \in D\}$ and $\lambda A := \{\lambda a \mid a \in A\}$.

The following properties will often be used in the sequel:

$$\lambda(A + D) = \lambda A + \lambda D, \quad \forall A, D \in 2^X$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu A, \quad \forall A \in 2^X,\ \lambda, \mu \in \mathbb{K}.$$

Also, if A is a convex set, then

$$(\lambda+\mu)A = \lambda A + \mu A, \quad \forall \lambda, \mu \in \mathbb{R},\ \lambda\mu \geq 0.$$

Moreover

$$\delta(A+D) \leq \delta(A) + \delta(D), \quad \forall A, D \in 2^X. \tag{7}$$

For more details and other relations see, e.g., [1,27,28].

In what follows, for each $D \subset X$, $cl(D)$ stands for the closure of the set D with respect to the norm in X. Moreover $a_{-1}(x) := 1$, $a_n(x) := \prod_{j=0}^{n} a(\varphi^j(x))$, $c_n(x) := b(\varphi^n(x))a_{n-1}(x)$, $s_{-1}(x) := 0$, and $s_n(x) := -\sum_{k=0}^{n} a_{k-1}(x)\psi(\varphi^k(x))$ for every $n \in \mathbb{N}_0$, $x \in S$.

The main result of this paper is contained in the next theorem.

Theorem 2.1. *Assume that $F : S \to 2^X$ is a set-valued map and the following three conditions hold:*

$$a(x)F(\varphi(x)) \subseteq F(x) + \psi(x) + b(x)B, \quad \forall x \in S, \tag{8}$$

$$\liminf_{n\to\infty} \delta(F(\varphi^{n+1}(x)))|a_n(x)| = 0, \quad \forall x \in S, \tag{9}$$

$$\omega(x) := \sum_{n=0}^{\infty} |c_n(x)| < \infty, \quad \forall x \in S. \tag{10}$$

Write

$$\Phi_n(x) := cl\left(a_{n-1}(x)F(\varphi^n(x)) + s_{n-1}(x) + \left(\sum_{k=n}^{\infty} |c_k(x)|\right) B\right)$$

for $x \in S$, $n \in \mathbb{N}_0$. Then, for each $x \in S$, the sequence $(\Phi_n(x))_{n\in\mathbb{N}_0}$ is decreasing (i.e., $\Phi_{n+1}(x) \subseteq \Phi_n(x)$), the set

$$\widehat{\Phi}(x) := \bigcap_{n=0}^{\infty} \Phi_n(x)$$

has exactly one point and the function $f : S \to X$ given by $f(x) \in \widehat{\Phi}(x)$ is the unique solution of Eq. (6) with

$$f(x) \in \Phi_0(x) = cl(F(x) + \omega(x)B), \quad \forall x \in S. \tag{11}$$

Proof. Fix $x \in S$. Replacing x by $\varphi^n(x)$ in (8) and multiplying both sides of the resulting inclusion by $a_{n-1}(x)$ we get

$$a_n(x)F(\varphi^{n+1}(x)) \subseteq a_{n-1}(x)[F(\varphi^n(x)) + \psi(\varphi^n(x)) + b(\varphi^n(x))B]$$

for each $n \in \mathbb{N}_0$, which can also be written in the form

$$a_n(x)F(\varphi^{n+1}(x)) + s_n(x) \subseteq a_{n-1}(x)F(\varphi^n(x)) + s_{n-1}(x) + c_n(x)B. \tag{12}$$

Let $r_n(x) := \sum_{k=n+1}^{\infty} |c_k(x)|$ for $x \in S$, $n \in \mathbb{N}_0 \cup \{-1\}$. In view of (10),

$$\lim_{n\to\infty} c_n(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} r_n(x) = 0. \tag{13}$$

Adding $r_n(x)B$ to both sides of (12), we get

$$a_n(x)F(\varphi^{n+1}(x)) + s_n(x) + r_n(x)B \subseteq a_{n-1}(x)F(\varphi^n(x)) + s_{n-1}(x) + r_{n-1}(x)B$$

for $n \in \mathbb{N}_0$, because B is convex and balanced (and therefore $c_n(x)B = |c_n(x)|B$). Hence, for each $x \in S$, $(\Phi_n(x))_{n\in\mathbb{N}_0}$ is a decreasing sequence of closed sets and, in view of (7), (9) and (13), $\liminf_{n\to\infty} \delta(\Phi_n(x)) = 0$. So, the set $\widehat{\Phi}(x)$ has exactly one element, according to Cantor's theorem for decreasing sequences of closed sets in a complete metric space. Clearly, (11) holds.

Next, we show that f is a solution of (6). To this end observe that

$$\begin{aligned} a(x)\Phi_n(\varphi(x)) &= a(x)cl[a_{n-1}(\varphi(x))F(\varphi(\varphi^n(x))) + s_{n-1}(\varphi(x)) + r_{n-1}(\varphi(x))B] \\ &\subseteq cl(a_n(x)F(\varphi^{n+1}(x)) + s_n(x) + r_n(x)B) + \psi(x) \\ &= \Phi_{n+1}(x) + \psi(x) \end{aligned}$$

for each $n \in \mathbb{N}_0$, because $a(x)B = |a(x)|B$. Consequently

$$\begin{aligned} a(x)f(\varphi(x)) &\in \bigcap_{n=0}^{\infty} a(x)\Phi_n(\varphi(x)) \\ &\subseteq \bigcap_{n=0}^{\infty} \Phi_{n+1}(x) + \psi(x) = \{f(x) + \psi(x)\}. \end{aligned}$$

It remains to show the uniqueness. To this end, suppose that $f_1, f_2 : S \to X$ are solutions to Eq. (6) and $f_k(x) \in cl(F(x) + \omega(x)B)$ for $x \in S$, $k = 1, 2$. From (6) we get $a_{n-1}(x)f_k(\varphi^n(x)) = f_k(x) - s_{n-1}(x)$ for $n \in \mathbb{N}_0$, $x \in S$, $k = 1, 2$, whence

$$\begin{aligned}\|f_1(x) - f_2(x)\| &= |a_{n-1}(x)|\|f_1(\varphi^n(x)) - f_2(\varphi^n(x))\| \\ &\le |a_{n-1}(x)|\delta(F(\varphi^n(x))) + |a_{n-1}(x)|\omega(\varphi^n(x))\delta(B) \\ &= |a_{n-1}(x)|\delta(F(\varphi^n(x))) + r_{n-1}(x)\delta(B).\end{aligned}$$

Hence, by (9) and (13), $f_1(x) = f_2(x)$ for $x \in S$. □

From Theorem 2.1 there follows, for instance, the next corollary.

Corollary 2.2. *Let $d : S \to \mathbb{K} \setminus \{0\}$, $\psi_0 : S \to X$, $b_0 : S \to [0, \infty)$, $F : S \to 2^X$,*

$$F(\varphi(x)) \subseteq d(x)F(x) + \psi_0(x) + b_0(x)B, \quad \forall x \in S, \tag{14}$$

$$\liminf_{n\to\infty} \frac{\delta(F(\varphi^{n+1}(x)))}{\prod_{j=0}^{n} |d(\varphi^j(x))|} = 0, \quad \forall x \in S, \tag{15}$$

and

$$\omega_0(x) := \sum_{n=0}^{\infty} \frac{b_0(\varphi^n(x))}{\prod_{j=0}^{n} |d(\varphi^j(x))|} < \infty, \quad \forall x \in S. \tag{16}$$

Then there exists a unique solution $f : S \to X$ of the functional equation

$$f(\varphi(x)) = d(x)f(x) + \psi_0(x) \tag{17}$$

with $f(x) \in cl(F(x) + \omega_0(x)B)$ for $x \in S$.

Proof. Since B is balanced, by (14) we have

$$\frac{1}{d(x)}F(\varphi(x)) \subseteq F(x) + \frac{\psi_0(x)}{d(x)} + \frac{b_0(x)}{|d(x)|}B, \quad \forall x \in S.$$

So, it is enough to use Theorem 2.1, with $\psi(x) := \frac{\psi_0(x)}{d(x)}$, $a(x) := \frac{1}{d(x)}$ and $b(x) := \frac{b_0(x)}{|d(x)|}$ for $x \in S$. □

Remark 2.3. The sum of two nonempty closed subsets of X is not necessarily a closed set, but if A is closed and D is compact, then $A+D$ is a closed set. Therefore, if B is a compact set and the values of F are closed sets, then in Theorem 2.1 (respectively, in Corollary 2.2) we get the statement with $f(x) \in F(x) + \omega(x)B$ for $x \in S$ (respectively, $f(x) \in F(x) + \omega_0(x)B$ for $x \in S$).

The next corollary contains a stability result for Eq. (6) that corresponds to [25, Theorem 2.1].

Corollary 2.4. *Let (10) be valid and $g : S \to X$ satisfy*

$$a(x)g(\varphi(x)) - g(x) - \psi(x) \in b(x)B, \quad \forall x \in S.$$

Then there exists a unique solution $f : S \to X$ of Eq. (6) with $f(x) - g(x) \in \omega(x)cl(B)$ for $x \in S$. Moreover, for each $x \in S$,

$$f(x) = \lim_{n\to\infty} [a_{n-1}(x)g(\varphi^n(x)) + s_{n-1}(x)]. \tag{18}$$

Proof. Let $F : S \to 2^X$ be given by $F(x) = \{g(x)\}$ for $x \in S$. Then (8) and (9) hold. According to Theorem 2.1, there is a unique solution $f : S \to X$ of (6) with $f(x) \in cl(F(x) + \omega(x)B) = g(x) + \omega(x)cl(B)$ for $x \in S$. Condition (18) follows from the form of $\widehat{\Phi}(x)$. □

Using an approach analogous to that of the proof of Corollary 2.2, we deduce from Corollary 2.4 the following stability result generalizing [25, Theorem 2.1].

Corollary 2.5. *Suppose that $d : S \to \mathbb{K} \setminus \{0\}$, $b_0 : S \to [0, \infty)$, $\psi_0, g : S \to X$,*

$$g(\varphi(x)) - d(x)g(x) - \psi_0(x) \in b_0(x)B, \quad \forall x \in S, \tag{19}$$

and (16) holds. Then there is a unique solution $f : S \to X$ of the functional equation (17) with $f(x) - g(x) \in \omega_0(x)cl(B)$ for $x \in S$.

Remark 2.6. It is easily seen that, for $B := \{x \in X \mid \|x\| \le 1\}$, Corollary 2.5 gives exactly the same result as [25, Theorem 2.1].

Remark 2.7. Let $0 \notin b(S)$ and $m \in \mathbb{N}_0$. Then, in the case where

$$\xi := \sup_{x\in S} \frac{\delta(F(\varphi^m(x)))|a_{m-1}(x)|}{b(x)} < \infty, \tag{20}$$

condition (10) implies (9); in fact, we have

$$\frac{\delta(F(\varphi^m(\varphi^n(x))))|a_{m-1}(\varphi^n(x))|}{b(\varphi^n(x))} b(\varphi^n(x))|a_{n-1}(x)| = \delta(F(\varphi^{n+m}(x)))|a_{n+m-1}(x)|$$

and therefore

$$\delta(F(\varphi^{n+m}(x)))|a_{n+m-1}(x)| \le \xi |c_n(x)|$$

for $x \in S$, $n \in \mathbb{N}_0$. In particular, (20) holds, with $m = 0$, when $\sup_{x\in S} \delta(F(x)) < \infty$ and $\inf_{x\in S} b(x) > 0$. Analogously, if

$$\eta := \sup_{x\in S} \frac{\delta(F(\varphi(x)))|d_{m-1}(\varphi(x))|}{b(\varphi^m(x))} < \infty \tag{21}$$

(where $d_{-1}(x) := 1$ and $d_k(x) := d(\varphi^k(x))d_{k-1}(x)$ for $k \in \mathbb{N}_0$, $x \in S$), then

$$\frac{\delta(F(\varphi(\varphi^n(x))))|d_{m-1}(\varphi(\varphi^n(x)))|}{b_0(\varphi^m(\varphi^n(x)))} \frac{b_0(\varphi^{n+m}(x))}{\prod_{j=0}^{n+m} |d(\varphi^j(x))|} = \frac{\delta(F(\varphi^{n+1}(x)))}{\prod_{j=0}^{n} |d(\varphi^j(x))|}$$

for $n \in \mathbb{N}_0$, $x \in S$, which means that (15) follows from (16). As in the case of condition (20), inequality (21) holds, with $m = 0$, when $\sup_{x\in S} \delta(F(x)) < \infty$ and $\inf_{x\in S} b(x) > 0$.

We finish the paper with an example of the application of Theorem 2.1 in the investigation of selections of set-valued maps $G : T \to 2^X$ satisfying the following linear inclusion in two variables:

$$G(x \star y) \subseteq aG(x) + bG(y) + c_0 + D, \tag{22}$$

where D is a fixed convex and nonempty subset of X, $c_0 \in X$, $\star : T^2 \to T$ is a binary operation (in a set $T \neq \emptyset$) that is square symmetric (i.e., $(x \star y) \star (x \star y) = (x \star x) \star (y \star y)$ for $x, y \in T$), and $a, b \in \mathbb{K}$. Clearly, the mapping $\rho : T \to T$, $\rho(x) := x \star x$ for $x \in T$, is an endomorphism of the groupoid $(T, \star)$, whence

$$\rho^n(x \star y) = \rho^n(x) \star \rho^n(y), \quad \forall x, y \in T,\ n \in \mathbb{N}_0. \tag{23}$$

Let $(P, +)$ be a commutative semigroup, $\alpha, \beta : P \to P$ be endomorphisms with $\alpha \circ \beta = \beta \circ \alpha$ (e.g., $\alpha = \beta^n$ with some $n \in \mathbb{N}_0$), and $\gamma_0 \in P$. Then $* : P^2 \to P$, given by $x * y := \alpha(x) + \beta(y) + \gamma_0$ for $x, y \in P$, is square symmetric. Therefore, the next two corollaries complement and/or correspond to Theorem 1.1 and some results in [14,8,24,16,15,17,10].

Corollary 2.8. *Let $\mathbb{K} = \mathbb{R}$, $a > 0$, $b > 0$, $a + b > 1$, $S \subseteq T$, $\rho(S) \subseteq S$, $G : S \to 2^X$ satisfy (22) for $x, y \in S$ with $x \star y \in S$, and $\delta(D) < \infty$. Suppose that $\liminf_{n\to\infty} (a + b)^{-n}\delta(G(\rho^n(x))) = 0$ and the set $G(x)$ is convex for each $x \in S$. Then there is a unique function $g : S \to X$ with*

$$g(x \star y) = ag(x) + bg(y) + c_0, \quad \forall x, y \in S,\ x \star y \in S, \tag{24}$$

and

$$g(x) \in cl(G(x) + (a + b - 1)^{-1}D), \quad \forall x \in S.$$

Proof. Write $F(x) := G(x) + (a + b - 1)^{-1}(c_0 + D)$ for $x \in S$. Then

$$\begin{aligned} F(x \star y) &\subseteq aG(x) + bG(y) + c_0 + D + \frac{1}{a+b-1}(c_0 + D) \\ &= aF(x) + bF(y), \quad \forall x, y \in S,\ x \star y \in S. \end{aligned} \tag{25}$$

Hence

$$\frac{1}{a+b} F(\rho(x)) \subseteq \frac{a}{a+b} F(x) + \frac{b}{a+b} F(x) = F(x), \quad \forall x \in S.$$

Let

$$\Phi_n(x) := (a + b)^{-n} cl(F(\rho^n(x))), \quad \forall x \in S,\ n \in \mathbb{N}_0.$$

Then, in view of Theorem 2.1 (with $\varphi := \rho$, $\psi(x) := 0$, $a(x) := (a+b)^{-1}$, and $B := \{0\}$), the set

$$\widehat{\Phi}(x) := \bigcap_{n=0}^{\infty} \Phi_n(x)$$

has exactly one point for each $x \in S$, the function $f : S \to X$, given by $f(x) \in \widehat{\Phi}(x)$ satisfies $f(x) \in cl\,(F(x))$ for $x \in S$, and

$$\Phi_{n+1}(x) \subseteq \Phi_n(x), \quad \forall x \in S,\ n \in \mathbb{N}_0.$$

Note that the last inclusion yields

$$\lim_{n\to\infty} \delta(\Phi_n(x)) = 0, \quad \forall x \in S. \tag{26}$$

Take $x, y \in S$ with $x \star y \in S$. Since, by (23) and (25),

$$\Phi_n(x \star y) \subseteq a\Phi_n(x) + b\Phi_n(x), \quad \forall n \in \mathbb{N}_0,$$

we have

$$f(x \star y) \in \bigcap_{n=0}^{\infty} \Phi_n(x \star y) \subseteq \bigcap_{n=0}^{\infty} (a\Phi_n(x) + b\Phi_n(y)) =: H.$$

Further, in view of (7) and (26),

$$\lim_{n\to\infty} \delta(a\Phi_n(x) + b\Phi_n(y)) = 0,$$

whence $\delta(H) = 0$, which means that $H = \{f(x \star y)\}$. Consequently

$$f(x \star y) = af(x) + bf(y),$$

because (in view of the definition of H)

$$af(x) + bf(y) \in a\widehat{\Phi}(x) + b\widehat{\Phi}(y) \subseteq a\Phi_n(x) + b\Phi_n(y)$$

for $n \in \mathbb{N}_0$ and therefore $af(x) + bf(y) \in H$.

Now, it is easy to check that $g := f - (a+b-1)^{-1}c_0$ satisfies (24) for $x, y \in S$ with $x \star y \in S$. To complete the proof it is enough to notice that the uniqueness of g follows from the statement concerning uniqueness in Theorem 2.1. Namely, observe that for each function $g : S \to X$ satisfying (24) for $x, y \in S$ with $x \star y \in S$, the function $f := g + (a+b-1)^{-1}c_0$ fulfils

$$(a+b)^{-1} f(\rho(x)) = f(x), \quad \forall x \in S. \quad \square$$

Corollary 2.9. *Let $\mathbb{K} = \mathbb{R}$, $a > 0$, $b > 0$, $a + b > 1$, $S \subseteq T$, $\rho(S) \subseteq S$, $g : S \to X$, and*

$$g(x \star y) - ag(x) - bg(y) - c_0 \in D, \quad \forall x, y \in S,\ x \star y \in S.$$

Then there exists a unique function $f : S \to X$ such that (24) holds and

$$f(x) - g(x) \in (a+b-1)^{-1} cl(D), \quad \forall x \in S.$$

Proof. Define $G : S \to 2^X$ by $G(x) := \{g(x)\}$ for $x \in S$ and use Corollary 2.8. $\square$

Remark 2.10. Since $\bullet : X^2 \to X$, $x \bullet y := ax + by + c_0$, is square symmetric, Eq. (24) can be considered to be an equation of homomorphism of the square symmetric groupoids $(T, \star)$ and $(X, \bullet)$.

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